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A Note on the Matrix Analysis of Dual Reciprocity Boundary Element Method

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Abstract—This paper presents a matrix analysis of the Dual Reciprocity Boundary Element Method (DRM) applying to several kinds of boundary value problems. This method demonstrates how to reduce the storage and computation for the Stiffness matrix and its component matrices.

Keywords—BEM, Dual reciprocity, Numerical method, Matrix analysis.

1. INTRODUCTION

The Dual Reciprocity Boundary Element Method [1,2] has already been proved to be a powerful alternative to the traditional Boundary Element Method [3,4] (BEM). It can solve boundary value problems involving domain integrations that the BEM cannot efficiently solve and use the same fundamental solution in its formulation for different problems. Its success can be seen in applications for linear [5–7] and nonlinear [1,8,9] heat conduction problems.

However, like BEM, DRM also has a drawback in its Stiffness Matrix [1,10,11]. This matrix is unsymmetric and fully populated. This is in contrast to that of Finite Element Method [12], which is banded and sparse. Furthermore, it is formulated from the calculation of a series of component matrices [1] and this tends to make it more complicated than that of the BEM.

Obviously, this drawback will increase the computations both for forming the stiffness matrix, which involves matrix inverse and multiplication, and for solving the algebraic equation, which has to face a fully populated coefficient matrix.

Therefore, from the viewpoint of computation, a full investigation must be conducted to analyze the stiffness matrix along with its component matrices, to make clear their properties, and to reduce the computation in forming this matrix and, hopefully, in solving the algebraic equation.

2. MATRIX ANALYSIS FOR DIFFUSION PROBLEMS

Let us begin our analysis with a Poisson equation

$$\nabla^2 u = b, \tag{1}$$

where b is the heat source which is a known constant or function.

Based on the principles of DRM, this equation can be formulated into the matrix form

$$\mathbf{Hu} - \mathbf{Gq} = \left(\mathbf{H\hat{U}} - \mathbf{G\hat{Q}}\right)\mathbf{F}^{-1}\mathbf{b} \triangleq \mathbf{Sb}. \tag{2}$$

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Here, the vectors and matrices in equation (2) are defined as:

$\mathbf{u} = \{u_i\}_{nd\times 1}$, u_i is the nodal value of u at node i;

$\mathbf{q} = \{q_i\}_{nd\times 1}$, $q_i = \left.\dfrac{\partial u}{\partial n}\right|_{\text{node } i}$;

$\mathbf{b} = \{b_i\}_{nd\times 1}$, b_i is the nodal value of source b at node i;

$$\mathbf{H} = \{H_{ij}\}_{nd\times nd}, \quad H_{ij} = \begin{cases} c_i\delta_{ij} + \displaystyle\sum_{IEN(a,e)=j} \int_{\Gamma^e} q^* N_a d\Gamma, & j \le nb; \\ 1, & nb+1 \le j \le nd, \quad i = j; \\ 0, & \text{otherwise}; \end{cases}$$

$$\mathbf{G} = \{G_{ij}\}_{nd\times nb}, \quad G_{ij} = \begin{cases} \displaystyle\sum_{IEN(a,e)=j} \int_{\Gamma^e} u^* N_a d\Gamma, & j \le nd; \\ 0, & nb+1 \le j \le nd; \end{cases}$$

$\mathbf{F} = \{f_{ij}\}_{nd\times nd}$, f_{ij} is the function value of f_j at node i;

$\mathbf{F}^{-1}$ inverse of $\mathbf{F}$;

$\hat{\mathbf{U}} = \{\hat{u}_{ij}\}_{nd\times nd}$, $\hat{u}_{ij}$ is the function value of $\hat{u}_j$ at node i;

$$\hat{\mathbf{Q}} = \{\hat{q}_{ij}\}_{nd\times nd}, \quad \hat{q}_{ij} = \begin{cases} \left.\dfrac{\partial \hat{u}_j}{\partial n}\right|_{\text{node } i}, & i \le nb; \\ 0, & nb+1 \le i \le nd. \end{cases}$$

where nd is the number of boundary and internal nodes, nb is the number of boundary nodes, a, e are the node and element number, $IEN(a,e)$ is the global node number of node a on element e, u^* is the fundamental solution to $\nabla^2 u = 0$, c_i is a geometry coefficient, δ_{ij} is the delta function, f_j is the approximation function, and $\hat{u}_j$ is the analytical solution to $\nabla^2 \hat{u}_j = f_j$.

Usually, we select the distance function as the approximation function $f_j = 1 + r_j$, where $r_j = r(\mathbf{x}, \mathbf{x}_j)$ is the distance between point $\mathbf{x}$ and $\mathbf{x}_j$. From this, we know that

$$\mathbf{F}^\top = \mathbf{F} \Rightarrow \left(\mathbf{F}^\top\right)^{-1} = \left(\mathbf{F}^{-1}\right)^\top. \tag{3}$$

Also,

$$\nabla^2 \hat{u}_j = f_j \Rightarrow \hat{u}_j = \frac{r_j^2}{6}(r_j + 3) \Rightarrow \hat{\mathbf{U}}^\top = \hat{\mathbf{U}}. \tag{4}$$

We can make use of the symmetry of $\mathbf{F}$ and $\mathbf{U}$ to reduce their storage space, the computation in calculating $\mathbf{F}^{-1}$, and the computation of matrix products relating to $\mathbf{F}$ and $\mathbf{U}$.

To matrix H and Q, from their definitions, we know that they involve these integrations

$$\begin{gathered} \int_{\Gamma^e} q^* N_a d\Gamma, \qquad \int_{\Gamma^e} u^* N_a d\Gamma \\ u^* = \frac{1}{4\pi r}, \qquad q^* = \frac{\partial u^*}{\partial n}, \qquad N_a = N_a(\xi, \eta) = \frac{1}{4}(1 + \xi_a \xi)(1 + \eta_a \eta). \end{gathered} \tag{5}$$

Even the constant element is selected and the interpolation function N_a is avoided, we still cannot integrate them analytically for the 3-D case and have to use the numerical method. Also, we know that H_{ij}, G_{ij} are boundary geometry dependent, therefore, matrix $\mathbf{H}$ and Q are unsymmetric and not fully populated.

For matrix $\hat{\mathbf{Q}}$, we know that

$$\hat{q}_j = \mathbf{n} \cdot \mathbf{r}_j \left(\frac{r_j}{2} + 1\right), \tag{6}$$

although $(r_j/2) + 1$ could lead to a symmetric matrix, the directional derivative term $\mathbf{n} \cdot \mathbf{r}_j$ prevents it to be symmetric, and we have to compute the whole $\hat{\mathbf{Q}}$.

Now, from its definition and its component matrices, we know that matrix $\mathbf{S}$ is unsymmetric and fully populated.

3. MATRIX ANALYSIS FOR CONVECTION-DIFFUSION PROBLEMS

Now let us go to the Convection-Diffusion problems governed by equation

$$\nabla^2 u + a_1 u_x + a_2 u_y + a_3 u_z + au = b, \tag{7}$$

where a_1, a_2, a_3 are the convection coefficients, a is the diffusion coefficient, b is the source. Following similar steps, we can get the matrix equation of the Convection-Diffusion problem

$$\mathbf{Hu} - \mathbf{Gq} + \mathbf{S}\left(\sum_{l=1}^{3} \mathbf{A}_l \mathbf{F}_{x_l} + \mathbf{A}\right)\mathbf{u} = \mathbf{Sb}, \tag{8}$$

where $\mathbf{A}_l, \mathbf{A}$ are diagonal matrices whose elements are nodal values of a_l, a, respectively, and $\mathbf{F}_{x_l}$ is the derivative of $\mathbf{F}$ with respect to x_l.

From the definition of f_j, we know that

$$f_j = 1 + r_j \Rightarrow \frac{\partial f_j}{\partial x_l} = \frac{x_l - x_{jl}}{r_j} \Rightarrow \mathbf{F}_{x_l}^{\mathsf{T}} = -\mathbf{F}_{x_l}, \tag{9}$$

and we define the diagonal element of $\mathbf{F}_{x_l}$ to be zero due to singularity. Therefore, we can also make use of the symmetry to reduce the storage space for $\mathbf{F}_{x_l}$, and the computation of matrix products in equation (8).

4. TRANSIENT PROBLEMS

Finally, we demonstrate the transient problems governed by equation

$$c_1 u_{tt} + c_2 u_t = \nabla^2 u + a_1 u_x + a_2 u_y + a_3 u_z + au + b, \tag{10}$$

where c_1, c_2 are known coefficients, a_1, a_2, a_3, a, and b are defined in previous section.

Here, if $c_1 = 0$ and $c_2 \neq 0$, it is a transient heat conduction equation; if $c_1 \neq 0$, it is a wave equation with or without a damping term depending on the value of c_2.

In the same manner, we get the matrix equation

$$\mathbf{C_1 S \ddot{u}} + \mathbf{C_2 S \dot{u}} = \mathbf{Hu} - \mathbf{Gq} + \mathbf{S}\left(\sum_{l=1}^{3} \mathbf{A}_l \mathbf{F}_{x_l} + \mathbf{A}\right)\mathbf{u} + \mathbf{Sb}, \tag{11}$$

where $\mathbf{C}_1, \mathbf{C}_2$ are diagonal matrices containing nodal values of c_1, c_2; $\ddot{\mathbf{u}}, \dot{\mathbf{u}}$ are vectors containing nodal values of u_{tt}, u_t.

At once, we can see that the matrices are the same as those in equation (8) except that $\mathbf{C}_1, \mathbf{C}_2$ are diagonal, which can also reduce the storage space and computation for matrix products.

5. STIFFNESS MATRIX

In Section 2, we can apply the boundary conditions and rearrange the known term to form the stiffness matrix. This matrix is constructed like this: for $0 \leq j \leq nb$, its column vectors are those of matrix $\mathbf{H}$ and $\mathbf{G}$ corresponding to the unknown values of $\mathbf{u}_j$ and $\mathbf{q}_j$; for $nb + 1 \leq j \leq nd$, it simply copies the column vectors of $\mathbf{H}$. From this, we know that the stiffness matrix is not fully populated and is unsymmetric.

In Section 3, we have to add matrix $\mathbf{H}$ and products relating to $\mathbf{S}$ together before constructing the stiffness matrix, it makes this matrix fully populated and unsymmetric.

In Section 4, in order to get the response at different time steps, we use the temporal difference to approximate the derivatives with respect to time and then apply the boundary conditions and rearrange the known terms to get the stiffness matrix. Because of the existence of matrix $\mathbf{S}$, we also get the same result of Section 3.

6. CONCLUSION

Throughout this analysis, we know that, although the Stiffness matrix of DRM is unsymmetric and fully populated, we can still reduce the storage space and computation for its component matrices, and hence, reduce the storage space and computation for the overall process.

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